11.4 - Jump Processes and Their Integrals

11.4.2 Quadratic Variation

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Quadratic Variation

Definition

- Let X(t) be a jump process.
- To compute the quadratic variation of X on [0, T],

  we choose $0 = t_0 < t_1 < t_2 < \ldots < t_n = T$, denote the set of these times by $\Pi = \{t_0, t_1, \ldots, t_n\}$, denote the length of the longest subinterval by

  $||\Pi|| = max_j(t_{j+1} - t_j)$, and define

$$Q_\Pi(X) = \sum_{j=0}^{n-1} (X(t_{j+1}) - X(t_j))^2$$

- The quadratic variation of X on [0, T] is defined to be

$$[X, X](T) = \lim_{||\Pi|| \to 0} Q_\Pi(X)$$

  where of course as $||\Pi|| \to 0$ the number of points in Π must approach infinity.

Quadratic Variation

Definition

- In general, [X, X](T) can be random (i.e., can depend the path of X)
- However, in the case of Brownian motion, we know that [W, W](T) = T
- In the case of an Itô integral $I(T) = \int_0^T \Gamma(s)dW(s)$ with respect to Brownian motion, $[I, I](T) = \int_0^T (\Gamma(s)dW(s))^2 = \int_0^T \Gamma(s)^2 dW(s)^2 = \int_0^T \Gamma^2(s)ds$ can depend on the path because $\Gamma(s)$ can depend on the path.

Quadratic Variation

Cross variation

- Let $X_1(t)$ and $X_2(t)$ be jump processes

$$C_\Pi(X_1, X_2) = \sum_{j=0}^{n-1} (X_1(t_{j+1} - X_1(t_j))(X_2(t_{j+1} - X_2(t_j))$$

and

$$[X_1, X_2](T) = \lim_{\|\Pi\| \to 0} C_\pi(X_1, X_2)$$

Quadratic Variation

Theorem 11.4.7

- Let $X_1(t) = X_1(0) + I_1(t) + R_1(t) + J_1(t)$ be a jump process, where

  $I_1(t) = \int_0^t \Gamma_1(s)dW(s)$, $R_1(t) = \int_0^t \Theta_1(s)ds$, $J_1(t)$ is a right-continuous pure jump process.

  Then $X_1^c(t) = X_1(0) + I_1(t) + R_1(t)$ and

  $[X_1, X_1](T) = [X_1^c, X_1^c](T) + [J_1, J_1](T) = \int_0^T \Gamma_1^2(s)ds + \sum_{0<s\leq T} (\Delta J_1(s))^2$ (11.4.11)

Quadratic Variation
Theorem 11.4.7

- Let $X_2(t) = X_2(0) + I_2(t) + R_2(t) + J_2(t)$ be another jump process, where

  $I_2(t) = \int_0^t \Gamma_2(s)dW(s)$, $R_2(t) = \int_0^t \Theta_2(s)ds$, $J_2(t)$ is a right-continuous pure jump process.

  Then $X_2^c(t) = X_2(0) + I_2(t) + R_2(t)$ and

  $$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s) \quad (11.4.12)$$

$X(t) = X(0) + I(t) + R(t) + J(t)$ (11.4.1)

$X^c(t) = X(0) + I(t) + R(t)$

$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\le T} \Delta J_1(s)\Delta J_2(s)$ (11.4.12)

Quadratic Variation

Theorem 11.4.7 PROOF

- Only need to prove (11.4.12), since (11.4.11) is the special case of (11.4.12) in which $X_2 = X_1$

$$C_\Pi(X_1, X_2) = \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j))$$

拆解 $$= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j) + J_1(t_{j+1}) - J_1(t_j)) \times (X_2^c(t_{j+1}) - X_2^c(t_j) + J_2(t_{j+1}) - J_2(t_j))$$

乘開 $$= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j))$$

$$+ \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(J_2(t_{j+1}) - J_2(t_j))$$

$$+ \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j))$$

$$+ \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j))(J_2(t_{j+1}) - J_2(t_j))$$ (11.4.13)

$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\le T} \Delta J_1(s)\Delta J_2(s) \quad (11.4.12)$$

Quadratic Variation

Theorem 11.4.7 PROOF

- $$\lim_{\|\Pi\|\to 0} \sum_{j=1}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j)) = [X_1^c, X_2^c](T)$$
  $$= \int_0^T \Gamma_1(s)\Gamma_2(s)ds$$

- $$[J_1, J_2](T) = \sum_{0<s\le T} \Delta J_1(s)\Delta J_2(s)$$

Quadratic Variation

Theorem 11.4.7 PROOF

$X(t) = X(0) + I(t) + R(t) + J(t)$ (11.4.1)

$X^c(t) = X(0) + I(t) + R(t)$

$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\le T} \Delta J_1(s)\Delta J_2(s)$ (11.4.12)

- Only need to prove (11.4.12), since (11.4.11) is the special case of (11.4.12) in which $X_2 = X_1$

$$
\begin{aligned}
C_\Pi(X_1, X_2) &= \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j)) \\
&= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j) + J_1(t_{j+1}) - J_1(t_j)) \times (X_2^c(t_{j+1}) - X_2^c(t_j) + J_2(t_{j+1}) - J_2(t_j)) \\
&= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j)) \\
&\quad + \sum_{j=0}^{n-1} \underline{(X_1^c(t_{j+1}) - X_1^c(t_j))(J_2(t_{j+1}) - J_2(t_j))} \\
&\quad + \sum_{j=0}^{n-1} \underline{(J_1(t_{j+1}) - J_1(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j))} \\
&\quad + \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j))(J_2(t_{j+1}) - J_2(t_j))
\end{aligned}
\qquad (11.4.13)
$$

$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s) \quad (11.4.12)$$

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Theorem 11.4.7 PROOF

$$\left| \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(J_2(t_{j+1}) - J_2(t_j)) \right|$$

$$\leq \max_{0\leq j\leq n-1} |X_1^c(t_{j+1}) - X_1^c(t_j)| \cdot \sum_{j=0}^{n-1} |J_2(t_{j+1}) - J_2(t_j)|$$

$$\leq \underbrace{\max_{0\leq j\leq n-1} |X_1^c(t_{j+1}) - X_1^c(t_j)|}_{\text{由於 } \|\Pi\| \to 0 \text{ 故該項有極限0}} \cdot \sum_{0<s\leq T} \underbrace{|\Delta J_2(s)|}_{\text{不仰賴}\Pi\text{的有限數}}$$

Similarly, the third term $\left| \sum_{j=0}^{n-1} (X_2^c(t_{j+1}) - X_2^c(t_j))(J_1(t_{j+1}) - J_1(t_j)) \right|$ has limit zero.

$$X(t) = X(0) + I(t) + R(t) + J(t) \quad (11.4.1)$$
$$X^c(t) = X(0) + I(t) + R(t)$$
$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\le T} \Delta J_1(s)\Delta J_2(s) \quad (11.4.12)$$

Quadratic Variation

Theorem 11.4.7 PROOF

- Only need to prove (11.4.12), since (11.4.11) is the special case of (11.4.12) in which $X_2 = X_1$

$$\begin{aligned}
C_\Pi(X_1, X_2) &= \sum_{j=0}^{n-1} (X_1(t_{j+1}) - X_1(t_j))(X_2(t_{j+1}) - X_2(t_j)) \\
&= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j) + J_1(t_{j+1}) - J_1(t_j)) \times (X_2^c(t_{j+1}) - X_2^c(t_j) + J_2(t_{j+1}) - J_2(t_j)) \\
&= \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j)) \\
&\quad + \sum_{j=0}^{n-1} (X_1^c(t_{j+1}) - X_1^c(t_j))(J_2(t_{j+1}) - J_2(t_j)) \\
&\quad + \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j))(X_2^c(t_{j+1}) - X_2^c(t_j)) \\
&\quad + \sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j))(J_2(t_{j+1}) - J_2(t_j))
\end{aligned} \quad (11.4.13)$$

$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq T} \Delta J_1(s)\Delta J_2(s) \quad (11.4.12)$$

Quadratic Variation

Theorem 11.4.7 PROOF

- Let us fix an arbitrary $\omega \in \Omega$, which fixes the paths of these processes, and choose the time points in Π so close together that there is at most one jump of J_1 in each interval $(t_j, t_{j+1}]$, at most one jump of J_2 in each interval $(t_j, t_{j+1}]$, and if J_1 and J_2 have a jump in the same interval, then these jumps are simultaneous.
- Let A_1 denote the set of indices j for which $(t_j, t_{j+1}]$ contains a jump of J_1, and let A_2 denote the set of indices j for which $(t_j, t_{j+1}]$ contains a jump of J_2.

$$\sum_{j=0}^{n-1} (J_1(t_{j+1}) - J_1(t_j))(J_2(t_{j+1}) - J_2(t_j))$$

$$= \sum_{j \in A_1 \cap A_2} (J_1(t_{j+1}) - J_1(t_j))(J_2(t_{j+1}) - J_2(t_j))$$

$$= \sum_{0<s\leq t} \Delta J_1(s)\Delta J_2(s)$$

~~t~~ T (應是作者筆誤)

$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\le T} \Delta J_1(s)\Delta J_2(s) \quad (11.4.12)$$

Quadratic Variation

Remark 11.4.8

- In differential notation, equation (11.4.12) of Theorem 11.4.7 says that if

$$X_1(t) = X_1(0) + X_1^c(t) + J_1(t), \quad X_2(t) = X_2(0) + X_2^c(t) + J_2(t),$$

應是作者筆誤 $X^c(t) = X(0) + I(t) + R(t)$

then

$$dX_1(t)dX_2(t) = dX_1^c(t)dX_2^c(t) + dJ_1(t)dJ_2(t)$$

In particular,

$$dX_1^c(t)dJ_2(t) = dX_2^c(t)dJ_1(t) = 0$$

- In order to get a nonzero cross variation, both processes must have a dW term or the processes must have simultaneous jumps

Quadratic Variation

Corollary 11.4.9

- Let $W(t)$ be a Brownian motion and $M(t) = N(t) - \lambda t$ be a compensated Poisson process relative to the same filtration $F(t)$ (Definition 11.4.1). Then

$$[W, M](t) = 0, \quad t \geq 0.$$

- **PROOF**

  In Theorem 11.4.7, take $I_1(t) = W(t), R_1(t) = J_1(t) = 0$ and take

  $I_2(t) = 0, \quad R_2(t) = -\lambda t, \quad$ and $\quad J_2(t) = N(t)$

$$[X_1, X_2](T) = [X_1^c, X_2^c](T) + [J_1, J_2](T) = \int_0^T \underbrace{\Gamma_1(s)\Gamma_2(s)ds}_{I_2(t)\text{為0，該項就為0}} + \sum_{0<s\leq T} \underbrace{\Delta J_1(s)\Delta J_2(s)}_{J_1(t)\text{為0，該項就為0}} \quad (11.4.12)$$

- **Meaning**

  In Corollary 11.5.3 that the equation [W, M](t) = 0 implies that W and M are independent, and hence W and N are independent.

  A Brownian motion and a Poisson process relative to the same filtration must be independent.

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Corollary 11.4.10

- For $i = 1,2$, let $X_i(t)$ be an adapted, right-continuous jump process.

  In other words, $X_i(t) = X_i(0) + I_i(t) + R_i(t) + J_i(t)$, where $I_i(t) = \int_0^t \Gamma_i(s)dW(s)$, $R_i(t) = \int_0^t \Theta_i(s)ds$, and $J_i(t)$ is a pure jump process.

- Let $\widetilde{X}_i(0)$ be a constant, let $\phi_i(s)$ be an adapted process, and set

$$\widetilde{X}_i(t) = \widetilde{X}_i(0) + \int_0^t \Phi_i(s)dX_i(s)$$

By definition,

$$\widetilde{X}_i(t) = \widetilde{X}_i(0) + \widetilde{I}_i(t) + \widetilde{R}_i(t) + \widetilde{J}_i(t)$$

where

$$\widetilde{I}_i(t) = \int_0^t \Phi_i(s)\Gamma_i(s)dW(s),\ \widetilde{R}_i(t) = \int_0^t \Phi_i(s)\Theta_i(s)ds,\ \widetilde{J}_i(t) = \sum_{0<s\leq t} \Phi_i(s)\Delta J_i(s)$$

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Corollary 11.4.10

- Note that $\widetilde{X}_i(t)$ is a jump process with continuous part $\widetilde{X}_i^c(t) = \widetilde{X}_i(0) + \widetilde{I}_i(t) + \widetilde{R}_i(t)$ and pure jump part $\widetilde{J}_i(t)$. We have

$$
\begin{aligned}
&[\widetilde{X}_1, \widetilde{X}_2](t) \\
&= [\widetilde{X}_1^c, \widetilde{X}_2^c](t) + [\widetilde{J}_1, \widetilde{J}_2](t) \\
&= \int_0^t \Phi_1(s)\Phi_2(s)\Gamma_1(s)\Gamma_2(s)ds + \sum_{0<s\leq t} \Phi_1(s)\Phi_2(s)\Delta J_1(s)\Delta J_2(s) \\
&= \int_0^t \Phi_1(s)\Phi_2(s)d[X_1, X_2](s)
\end{aligned}
$$

Quadratic Variation

Remark 11.4.11

- Corollary 11.4.10 may be rewritten using differential notation.
- The corollary says that if

$$d\widetilde{X}_1(t) = \Phi_1(t)dX_1(t) \text{ and } d\widetilde{X}_2(t) = \Phi_2(t)dX_2(t)$$

then

$$d\widetilde{X}_1(t)d\widetilde{X}_2(t) = \Phi_1(t)\Phi_2(t)dX_1(t)dX_2(t)$$

Thanks for listening